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Note

Characterization of graphs with equal bandwidth and cyclic bandwidth ☆

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Abstract

$B(G)$ and $B_c(G)$ denote the bandwidth and cyclic bandwidth of graph G, respectively. In this paper, we shall give a characterization of graphs with equal bandwidth and cyclic bandwidth. Those graphs include any plane graph G with $B(G) < p/m$, where p and m are the number of vertices and the maximum degree of bounded faces of G, respectively. Hence convex triangulation meshes $T_{m,n,l}$ with $\min\{m,n,l\}\geqslant 4$ and grids $P_m \times P_n$ with $m\geqslant 3$ also fall in this class.
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1. Introduction

In this paper, $G=(V,E)$ shall be a graph of order p. A one-to-one mapping from V onto $\{1,2,\ldots,p\}$ is called a *numbering* of G.

Definition 1.1. Suppose f is a numbering of G. Let $B(G,f)=\max_{uv\in E}|f(u)-f(v)|$. The bandwidth of G, denoted by $B(G)$, is

$$\min_f\{B(G,f)\colon f \text{ is a numbering of } G\}.$$

A numbering f of G satisfying $B(G)=B(G,f)$ is called an optimal numbering of G.

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Definition 1.2. Suppose f is a numbering of G. Let $B_c(G,f) = \max_{uv\in E} ||f(u) - f(v)||_c$, where $||x||_c = \min\{|x|,\ p - |x|\}$ for $0 < |x| < p$. The cyclic bandwidth of G, denoted by $B_c(G)$, is defined as

$$B_c(G) = \min_f \{B_c(G,f)\text{: } f \text{ is a numbering of } G\}.$$

A numbering f of G satisfying $B_c(G) = B_c(G,f)$ is called a cb-optimal numbering of G.

The bandwidth problem of graphs has a wide range of applications including sparse matrix computation, data structure, coding theory and circuit layout of VLSI designs (see [6]). The problem became very important since the mid-1960s — see [2] or [3]. In its original formulation, the problem is to lay vertices of a graph on a path in such a way so that the maximum distance between any two vertices connected by an edge is minimized. Besides a path, other candidates are also available, and at times may even be more appropriate. In [6,12], laying vertices on grids $P_m \times P_n$ (product of two paths) and on a cycle C_n, respectively, are considered. When vertices are laid on a cycle, we get cyclic bandwidth (Definition 1.2), which we shall study in this paper.

For a graph G in general, $B_c(G) \leqslant B(G) \leqslant 2B_c(G)$, and both bounds are sharp. In [10], we obtained a sufficient condition for a graph to have equal bandwidth and cyclic bandwidth, namely graphs without *long cycles*. However, many graphs possess long cycles and yet their bandwidth and cyclic bandwidth are equal. For example, let G be a graph consisting of a cycle C and a vertex v which does not belong to the cycle, but is adjacent to one of the vertices of the cycle. In this paper, we shall give a necessary and sufficient conditons for a graph to have equal bandwidth and cyclic bandwidth.

In Sections 2 and 3, we introduce the concept of *zero/non-zero cycles* and *proper realignment*, respectively. In Section 4, we use these concepts to show that bandwidth is equal to cyclic bandwidth for a graph with a cb-optimal numbering containing no non-zero cycles. Finally, we show also that convex triangulation meshes $T_{m,n,l}$ with $\min\{m,n,l\} \geqslant 6$ and grids $P_m \times P_n$ with $m \geqslant 5$ fall in this class. For notation and terminology of graph theory, please refer to the book of Bondy and Murty [1] and Grimaldi [5] unless defined otherwise.

2. Zero and non-zero cycles

Definition 2.1. Let f be a numbering of G. For any $u, v \in V$ such that $uv \in E$, the cyclic displacement of the numbering f from u to v, denoted by $d_f(u,v)$, is $f(v) - f(u) + p\delta_{v,u}$, where

$$\delta_{v,u} = \begin{cases} 0 & \textit{if } |f(v) - f(u)| \leqslant p/2, \\ 1 & \textit{if } f(v) - f(u) < -\,p/2, \\ -1 & \textit{if } f(v) - f(u) > p/2. \end{cases}$$

Note that $||f(v) - f(u)||_c = |d_f(u,v)|$.

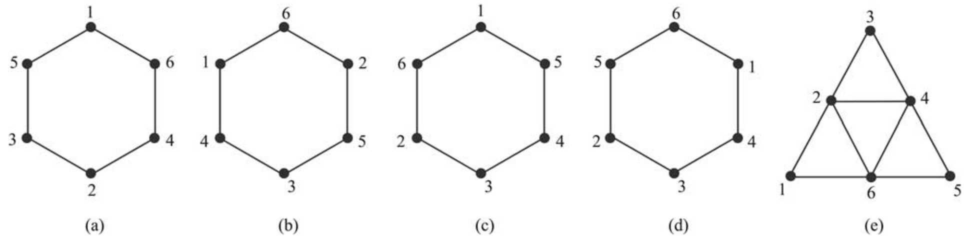

Fig. 1.

Definition 2.2. Let f be a numbering of G and C: $v_1v_2\dots v_kv_{k+1}=v_1$ a cycle in G. The total cyclic displacement of the numbering f on C, denoted by S_C, is the sum of cyclic displacements of edges in C.

It is easy to see that $S_C=\lambda p$, where λ is an integer. We call the cycle C a *zero cycle* of f if $\lambda=0$; otherwise, we call C a *non-zero cycle* of f. For examples, the 6-cycle is a zero cycle of the numberings indicated in Figs. 1(a) and (b), and is a non-zero cycle of the numberings indicated in Figs. 1(c) and (d).

3. Proper realignment

Definition 3.1. Suppose f is a numbering of G. A one-to-one mapping g from V into $\mathbb{N}$ is called a *proper realignment* of f if

$$|g(v)-g(u)|\leqslant\|f(v)-f(u)\|_c,\quad \text{for any } uv\in E.$$

The following lemma on proper realignment can be found in [10].

Lemma 3.2. *Suppose f is a numbering of a tree T. Then there exists a proper realignment of f.*

We can construct a proper realignment of f by the following steps:

1. Choose a vertex $v\in V$. Set $S=\{v\}$ and put $g(v)=f(v)$.
2. $T[S]$ is a tree. For any $v\in N(S)$, there exists $u\in S$ which is adjacent to v. This u is also unique, because T, being a tree, contains no cycles and two vertices in S cannot be both adjacent to v. Put $g(v)=g(u)+d_f(u,v)$.
3. Put $S=S\cup\{v\}$. If $S\neq V$, then go to (2). Otherwise, stop.

The remark follow Lemma 3.2:

Remark 3.1. (1) If u and v are two vertices in $V[T]$, then $g(u)\neq g(v)$.
(2) If $vu\in E[T]$, then $g(v)-g(u)=d_f(u,v)$.

4. Characterization of graphs with equal bandwidth and cyclic bandwidth

Theorem 4.1. *Suppose G is a graph. There exists a cb-optimal numbering f of G containing no non-zero cycles if and only if $B_c(G) = B(G)$.*

Proof. Suppose G is a graph and f is a cb-numbering of G containing no non-zero cycles. We take an arbitrary spanning tree T from G and then construct a proper realignment of f by the Proper Realignment Algorithm. For any $vu \in E[T]$, it is clear that $|g(v) - g(u)| \leqslant B_c(T, f) \leqslant B_c(G, f) = B_c(G)$. If we can also show that $|g(v) - g(u)| \leqslant B_c(G)$ for any $vu \in E[G] \setminus E[T]$, then we have $B_c(G) = B(G)$.

Now let $e = vu \in E[G] \setminus E[T]$ and $C = u_1 u_2 \dots u_m u_{m+1}$, where $u_1 = u_{m+1} = v$ and $u_m = u$, be a cycle in $E[T] + e$. Then from Remark 3.1(2) in Section 3, we have

$$S_C = \sum_{i=1}^{m} d_f(u_i, u_{i+1}) = d_f(u, v) + g(u) - g(v).$$

Since all cycles in G are zero cycles, therefore $S_C = 0$ and

$$|g(u) - g(v)| = |-d_f(u, v)| \leqslant B_c(G)$$

Conversely, suppose h is an optimal numbering of G and $B(G) = B_c(G)$. Since $B_c(G) \leqslant p/2$, we have $|h(v) - h(u)| \leqslant p/2$ and hence $d_h(u, v) = h(v) - h(u)$ for any $uv \in E$. Moreover, $||h(v) - h(u)||_c = |d_h(u, v)| = |h(v) - h(u)| \leqslant B(G) = B_c(G)$. Therefore, h is also a cb-optimal numbering of G. Also, for any cycle C: $v_1 v_2 \dots v_n v_{n+1}$ in G, where $v_{n+1} = v_1$, we have

$$S_C = \sum_{i=1}^{n} d_h(v_i, v_{i+1}) = \sum_{i=1}^{n} h(v_{i+1}) - h(v_i) = 0.$$

So h is a cb-optimal numbering of G containing no non-zero cycles. □

Because trees are acyclic, we obtain the following result of [10] from Theorem 4.1 as a corollary.

Corollary 4.2. *If T is a tree, then $B_c(T) = B(T)$.*

It is known that the problem of determining the bandwidth of a graph is NP-complete even when it is restricted to trees with maximum degree three [6]. Therefore the following corollary, a main result of [12], holds.

Corollary 4.3. *The problem of determining the cyclic bandwidth of a graph is* NP-*complete.*

5. Graphs with equal bandwidth and cyclic bandwidth

Because the problem of determining the cyclic bandwidth of a graph is NP-complete, it is in general very difficult to obtain a cb-optimal numbering of a given graph G, not to mention the requirement of containing no non-zero cycles. However, in this section, we demonstrate that in some graphs, in addition to trees, a cb-optimal numbering containing no non-zero cycles exists. So Theorem 4.1 is applicable to some graphs containing cycles.

Lemma 5.1. *Suppose G is a graph and f is a numbering of G. If there exists a non-zero n-cycle of f in G, then $nB_c(G,f) \geqslant p$.*

Proof. Let C: $v_1v_2\dots v_nv_{n+1}$, where $v_{n+1}=v_1$, be a non-zero cycle in G of f. Then

$$p \leqslant |S_C| \leqslant \sum_{i=1}^{n} |d_f(v_i,v_{i+1})| = \sum_{i=1}^{n} ||f(v_i)-f(v_{i+1})||_c \leqslant nB_c(G,f). \qquad \square$$

Given a cycle C of a plane graph G, an edge is called an *internal edge* of C if it lies inside C. A path is called an *internal path* if it consists of internal edges of C solely.

Lemma 5.2. *Suppose G is a plane graph and f is a numbering of G. If the maximum degree of bounded faces of G is not greater than m, then either all cycles are zero cycles of f, or there exists a non-zero cycle of f with length m or less.*

Proof. Suppose C: $u_1u_2\dots u_lu_{l+1}$, where $u_{l+1}=u_1$, is a non-zero cycle of f enclosing k faces. If $k=1$, or if $k\geqslant 2$ and there is no internal path joining any two vertices of C, then clearly $l \leqslant m$.

Suppose $k\geqslant 2$ and there is an internal path $u_1v_2\dots v_mu_i$ joining u_1 to u_i, where $2\leqslant i\leqslant l$. Consider the two cycles C': $u_1v_2\dots v_mu_iu_{i+1}\dots u_lu_{l+1}$ and C^*: $u_1u_2\dots u_iv_mv_{m-1}\dots v_2u_1$. Noting that $d_f(u,v)=-d_f(v,u)$, we can show that

$$S_C = S_{C'} + S_{C^*}.$$

Since $S_C \neq 0$, therefore either $S_{C'} \neq 0$ or $S_{C^*} \neq 0$. In either case, we get a non-zero cycle of f enclosing at most $k-1$ faces. This process can continue until we get a non-zero cycle of f enclosing 1 face or having no internal paths joining any two vertices of the cycle. $\square$

Theorem 5.3. *Suppose G is a plane graph and the maximum degree of bounded faces is not greater than m. If $B(G)\leqslant \lceil p/m \rceil$, then $B_c(G)=B(G)$.*

Proof. Suppose $B_c(G) < B(G)$ and f is a cb-optimal numbering of G. By Theorem 4.1 and Lemma 5.2, f contains a non-zero cycle of length m or less. By Lemma 5.1,

$mB_c(G) \geqslant p$. It follows that $p/m \leqslant B_c(G)$ and consequently $\lceil p/m \rceil < B(G)$. The contradiction shows that $B_c(G) = B(G)$. □

Definition 5.4. A plane graph G whose bounded faces are all of degree m is called an m-gonal graph. If $m = 3$, G is called a triangulated graph.

Theorem 5.5. *Suppose G is an m-gonal graph with* $B(G) \leqslant \lceil p/m \rceil$. *Then* $B_c(G) = B(G)$.

Corollary 5.6. *If G is a triangulated graph and* $B(G) \leqslant p/3$, *then* $B_c(G) = B(G)$.

The product of two paths P_m and P_n is called an *mn-grid*. Because the bandwidth of an mn-grid is $\min\{m, n\}$ by [4], the following corollary holds.

Corollary 5.7. *If G is an mn-grid with* $n \geqslant m \geqslant 3$, *then* $B_c(G) = B(G)$.

The definition of a convex triangulation mesh $T_{m,n,l}$ was given in [11]. Because the bandwidth of a convex triangulation mesh $T_{m,n,l}$ is $\min\{m, n, l\}$ by [7,11], the next corollary follows.

Corollary 5.8. *For all convex triangulation meshes* $T_{m,n,l}$ *with* $\min\{m, n, l\} \geqslant 4$, *we have* $B_c(T_{m,n,l}) = B(T_{m,n,l})$.

Note that $B_c(T_{m,n,l}) \neq B(T_{m,n,l})$ if $m = n = l = 3$. The numbering of $G^* = T_{3,3,3}$ indicated in Fig. 1(e) shows that $B_c(G^*) \leqslant 2$, whereas $B(G^*) = 3$ by [7].

6. Uncited references

[8,9,13]

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